JHEP HYPER VERSION

Received: *November 23, 1998*, Accepted: *January 9, 1999*

AdS_2 quantum gravity and string theory

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Abstract: AdS_2 has an $SL(2,R)$ isometry group. It is argued that in the context of quantum gravity on AdS_2 this group is enlarged to the full infinite-dimensional $1+1$ conformal group. Massive scalar fields are coupled to AdS_2 gravity and shown to have associated conformal weights $h(m)$ shifted by their mass. For integral values of h primary boundary operators are obtained as h normal derivatives of the scalar field. AdS_2 string theories arise in the 'very-near-horizon' limit of S^1-compactified AdS_3 string theories. This limit corresponds to energies far below the compactification scale. The dual conformal field theory has one null dimension and can in certain cases be described as the discrete light cone quantization of a two-dimensional deformed symmetric-product conformal field theory. Evidence is given that the AdS_2 Virasoro algebra is related to the right-moving AdS_3 Virasoro algebra by a topological twist which shifts the central charge to zero.

Keywords: Field Theories in Lower Dimensions, Black Holes in String Theory, 2D Gravity.

JHEP01(1999)008

Contents

1. Introduction

Perhaps the best understood of the family of AdS/CFT dualities [1] is the case AdS_3/2D-CFT [2, 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], largely because the conformal group is infinite dimensional and greatly constrains the dynamics. In higher dimensions the field theory side is less well understood, and the duality has largely been used to learn about field theory from gravity. In lower dimensions — namely the AdS_2 case — one confronts the puzzles of black holes undistracted by inessential complications. Here one hopes to learn about gravity from field theory, and new wrinkles are encountered. One such wrinkle is that AdS_2 has two timelike boundaries, one originating from the region inside and one from the region outside the horizon. Hence the dual one-dimensional conformal quantum mechanics does not appear to live on a connected manifold.

At present very little is understood about this AdS_2 case, which is the subject of this paper. Some previous investigations appear in [13]. We first consider general properties of two-dimensional quantum gravity plus matter on AdS_2. While the subject of two-dimensional quantum gravity has received considerable attention[1], we shall see that AdS_2 provides an interesting new context. We then consider properties and dualities of AdS_2 string theories obtained by compactification of the well — understood AdS_3 case.

All theories of quantum gravity in two dimensions are conformal field theories [15, 16, 17]. Liouville-like theories with AdS_2 ground states — which necessarily are open-string-like theories on the strip — are no exception. A key distinction of the AdS_2 case discovered in [18, 19] is that, unlike on the circle or the plane, the theory on the strip has a ground state which is annihilated by the global $SL(2,R)$ subgroup of the conformal group. This subgroup is identified with the AdS_2 isometry group. We accordingly argue in section 2.1 that the global $SL(2,R)$ conformal symmetry in the AdS_2/CFT_1 duality is enlarged to the full local conformal group, much as in the AdS_3/CFT_2 case [2] . In section 2.2 we consider a scalar field ϕ of mass m coupled to gravity on AdS_2. It is shown perturbatively that the effect of the mass term is to shift the dimension of the highest weight state created by the scalar field to $h = \frac{1}{2}(1+\sqrt{1+m^2\ell^2})$, where ℓ is the AdS_2 radius. The corresponding boundary operator for integral h is $(\partial_\sigma)^h\phi$, with ∂_σ the normal derivative. This is primary because lower derivatives of ϕ vanish on the boundary. Section 2.3 contains a discussion of deficit angles produced by boundary insertions and a consequent upper bound on allowed masses.

In section 3 we turn to string theory. The near-horizon geometry of the five-dimensional black hole of [20] is AdS_3 periodically identified on a circle of asymptotic radius R [21]. In section 2.1 the interpretation of this as a compactification to two dimensions is presented. In section 3.2 it is shown that the geometry of the 'very-near-horizon' limit, corresponding to energies much less than $\frac{1}{R}$, is AdS_2 with a constant Kaluza-Klein $U(1)$ field strength. It is also found that in this limit the AdS_3 identification lies entirely within the $SL(2,R)_L$ subgroup of the $SL(2,R)_L \times SL(2,R)_R$ isometry group. The dual description of the very-near-horizon limit is argued in section 3.3 to be the DLCQ of the deformed symmetric product field theory described in [20]. Hence in this context the one dimension of the conformal field theory in the $AdS_2 \leftrightarrow CFT_1$ duality is null. This is naturally interpreted as half of a 2D conformal field theory rather than as conformal quantum mechanics. In section 3.4 the two-dimensional effective action arising from compactification of AdS_3 is described.

Symmetries are discussed in section 4. The AdS_3 string theory admits the action of two Virasoro algebras with large central charges. Only the right action survives the compactification. AdS_2 on the other hand admits the action of a single Virasoro algebra which has $c = 0$ since it is a theory of gravity. In section 4.1 we argue,

[1] A review with emphasis relevant to the present context can be found in the first few sections of reference [14].

assuming the existence of a suitable Hilbert space on AdS_2, that the AdS_2 Virasoro algebra is related to the AdS_3 algebra by a twisting which shifts the central charge to zero. This twisting is similar or identical to that used in earlier constructions of topological field theories and topological gravity from ordinary supersymmetric conformal field theories [22]. In section 4.2 the analysis of neutral scalar fields of section 2.2 is extended to the $U(1)$ charged scalar fields arising in AdS_3 compactification. It is found that due to cancellations the lowest conformal weight of the states created by such fields is independent of the charge and the Kaluza-Klein mass correction, and hence can be small even for very massive fields.

It should be noted AdS_2 string theories arise as the near-horizon limits of a wide variety of four and five -dimensional black holes without passing through an intermediate AdS_3 region. It is not clear to us which of the features discovered in sections 3 and 4 are generic to all AdS_2 string theories. The discussion of section 2 on the other hand, and in particular the enlargement of the $SL(2,R)$ to the full conformal group, pertains to all theories of quantum gravity on AdS_2.

2. AdS_2 quantum gravity as conformal field theory

In this section we consider general properties of two-dimensional quantum gravity coupled to matter with an AdS_2 ground state. We take Newton's constant to be small and the AdS_2 radius large so that a semiclassical description is appropriate. Since AdS_2 has two timelike boundaries, such theories live on the strip.[2] String theory on $AdS_2 \times M^8$ can be viewed as two-dimensional gravity with an infinite tower of fields arising from massive string and Kaluza-Klein modes. In conformal gauge

$$ds^2 = -e^{2\rho} dt^+ dt^- \,, \tag{2.1}$$

every two-dimensional quantum theory of gravity is a $c = 0$ (including ghosts) conformal field theory, with conformal invariance arising as a result of diffeomorphism invariance and background independence. Hence AdS_2 string theory can be viewed as conformal field theory on the strip. This is a lower dimensional analog of the philosophy of [2, 3] in which AdS_3 string theory is viewed as conformal field theory on the circle. Since the conformal transformations are residual diffeomorphisms, not all of the states in the theory should be regarded as inequivalent physical states. In a BRST formalism physical states will be identified as usual as BRST cohomology classes. In a ghost-free gauge one has a physical state condition of the form $L_n|\psi>= 0$. These are the quantum versions of the constraints which determine the metric in terms of the matter fields.

[2]We will mainly consider the covering space of AdS_2, which for brevity will be referred to simply as AdS_2.

2.1 AdS_2 isometries and the Virasoro algebra

The gravitational degrees of freedom of AdS_2 quantum gravity in conformal gauge are described by a Liouville theory governing the metric conformal factor ρ, possibly generalized by dilaton couplings. The action for a specific case in string theory is given in section 3 below. AdS_2 quantum gravity is like an open string theory on the strip owing to the two boundaries of AdS_2. Hence there is only one copy of the Virasoro algebra. This has an $SL(2,R)$ subgroup which generates the usual transformations

$$\begin{aligned}
L_{-1} &= \frac{i}{2}\left(e^{-2iu^+}\partial_+ + e^{-2iu^-}\partial_-\right)\,, \\
L_0 &= \frac{i}{2}(\partial_+ + \partial_-)\,, \\
L_1 &= \frac{i}{2}\left(e^{2iu^+}\partial_+ + e^{2iu^-}\partial_-\right)\,,
\end{aligned} \tag{2.2}$$

where $u^\pm = \frac{1}{2}(\tau \pm \sigma)$ and $0 \le \sigma \le \pi$. As noted some time ago by D'Hoker and Jackiw [18, 19],[3] the Liouville theory with cosmological constant $-\frac{2}{\ell^2}$ has an $SL(2,R)$-invariant vacuum characterized by

$$\ell e^{-\rho} = \sin(u^+ - u^-)\,. \tag{2.3}$$

This vacuum is of course the AdS_2 spacetime. The $SL(2,R)$ generators (2.2), originally introduced without reference to a metric, are the $SL(2,R)$ Killing vectors of AdS_2.

This situation contrasts with Liouville theory on the circle or plane, for which there is no vacuum preserving the $SL(2,R)_R \times SL(2,R)_L$ Virasoro subgroup except for the singular configuration $e^\rho = 0$. The absence of such an invariant vacuum has greatly complicated the analysis of the theory on the circle. Hence in this sense Liouville theory lives more naturally on the strip than on the plane.

An important conclusion here is that the global $SL(2,R)$ symmetry of AdS_2 string theory is enhanced to (half of) the $1+1$ conformal group, much as the global $SL(2,R)_L \times SL(2,R)_R$ symmetry of AdS_3 string theory is enhanced to the full $1+1$ conformal group. This conclusion follows from general properties of AdS_2 quantum gravity, and so applies to all AdS_2 string theories.

2.2 Coupling to massive scalars

Two-dimensional gravity may also contain scalar fields with actions of the form

$$S_\phi = -\frac{1}{4\pi}\int d^2x\sqrt{-g}\left((\nabla\phi)^2 + m^2\phi^2\right) = \frac{1}{8\pi}\int d^2u\left(4\partial_+\phi\partial_-\phi - m^2 e^{2\rho}\phi^2\right). \tag{2.4}$$

[3]The theory considered in [18, 19] differs in the scalar couplings from the string theory case considered below but this is not relevant for the following.

JHEP01(1999)008

We wish to study the dynamics of such scalar fields in a perturbation expansion about the AdS_2 vacuum (2.3). From the kinetic term we see that ϕ has global scaling dimension zero. The mass term is nevertheless a marginal perturbation because $e^{2\rho}$ is dimension $(1,1)$ operator.[4] The wave equation

$$\nabla^2\phi = m^2\phi\,, \tag{2.5}$$

reduces in conformal gauge to

$$-4e^{-2\rho}\partial_+\partial_-\phi = m^2\phi\,. \tag{2.6}$$

It is easily verified the AdS_2 laplacian defined by the metric (2.3) is precisely $-\frac{4}{\ell^2}$ times the $SL(2,R)$ Casimir obtained by squaring (2.2). Hence

$$-4(-L_0(L_0-1)+L_{-1}L_1)\phi = m^2\ell^2\phi\,. \tag{2.7}$$

For primary, normalizable solutions ϕ_h of (2.7) with $L_1\phi_h = 0$ and $L_0 = h$

$$4h(h-1)\phi_h = m^2\ell^2\phi_h \tag{2.8}$$

or

$$h = \frac{1}{2}\left(1+\sqrt{1+m^2\ell^2}\right), \tag{2.9}$$

in agreement with [24, 25, 4]. $L_1\phi_h = 0$ then implies

$$\phi_h = \left(e^{-2iu^+} - e^{-2iu^-}\right)^h\,. \tag{2.10}$$

Further solutions of (2.7) are generated by acting on ϕ_h with L_{-1}, as this commutes with the $SL(2,R)$ Casimir operator:

$$B_n(\sigma,\tau) = b_n(\sigma)e^{-i(n+h)\tau} = C_n(L_{-1})^n\phi_h\,, \qquad n\geq 0\,. \tag{2.11}$$

Together with their complex conjugates B_n^* these are a complete set of modes. We choose the normalization constants C_n so that

$$\langle B_m|B_n\rangle = \delta_{mn}\,, \tag{2.12}$$

with respect to the Klein-Gordon inner product

$$\langle A|B\rangle = -i\int_0^\pi d\sigma(A^*\partial_\tau B - \partial_\tau A^* B)\,. \tag{2.13}$$

ϕ may then be promoted to a quantum field and expanded in an oscillator basis

$$\phi = \sum_{n=0}^{\infty} a_n^\dagger B_n^* + a_n B_n\,. \tag{2.14}$$

[4] An explicit demonstration in a context close to the present one can be found in [23].

The canonical commutation relations for ϕ then imply as usual

$$\left[a_m, a_n^\dagger\right] = \delta_{mn} \,. \tag{2.15}$$

Using the ϕ stress tensor derived from the action (2.4), one finds that the quantum Virasoro generator L_0^ϕ gets a contribution

$$L_0^\phi = \sum_{n=0}^{\infty} (h+n) a_n^\dagger a_n \,. \tag{2.16}$$

It follows that, as expected, $a_n^\dagger$ creates a state with $L_0^\phi = h + n$. It is also possible to see that $[L_1^\phi, a_0^\dagger] = 0$ so that $a_0^\dagger|0\rangle$, where $|0\rangle$ is defined by $a_n|0\rangle = 0$, is a highest weight state.[5]

Therefore for every field of mass m there is a highest weight state of weight (2.9) in the conformal field theory. In conformal field theory on the strip there is a one-to-one correspondence[6] between highest weight states and primary operators at the boundary. For integral h the primary operator corresponding to $a_0^\dagger|0\rangle$ is

$$(\partial_\sigma)^h \phi(0,\tau) \,. \tag{2.17}$$

Ordinarily an operator such as (2.17) would not be primary due to the appearance of singular terms with lower derivatives of ϕ in the OPE with the stress tensor. However the mode expansion (2.14) for ϕ implies

$$(\partial_\sigma)^k \phi(0,\tau) = 0 \,, \qquad \text{for } k < h \,, \tag{2.18}$$

so that (2.17) is indeed primary.

2.3 A bound on the mass

The analysis of the previous section was relevant for small dimensionless Newton's constant G_N and $m^2\ell^2$ and h of order one. In string theory or Kaluza-Klein theory one encounters arbitrarily large values of m and so one is forced to consider the case $h \sim \frac{1}{G_N}$. In this case conformal invariance in a semiclassical expansion requires that (2.17) is replaced by the dressed operator[7]

$$e^{-h\rho(0,\tau)} (\partial_\sigma)^h \phi(0,\tau) \,. \tag{2.19}$$

For $h \sim \frac{1}{G_N}$, the semiclassical saddle point must include the ρ insertion on the boundary. This problem was considered in [26] (the bulk problem is considered

[5]The Virasoro constraints dictate the gravitational dressing of this state, which is higher order in the perturbative regime of small Newton's constant considered here.

[6]Modulo possibly relevant subtleties discussed in [14].

[7]The coefficient of ρ in the exponent will be altered if there are leading semiclassical corrections to the dimension of $e^{-h\rho}$. This is indeed the case for the pure Liouville theory considered in [26, 14].

in [14]) where it was shown semiclassically that the corresponding operator insertions produce kinks or deficit angles in the boundary of order hG_N. This follows from the ρ equation of motion which has a delta function source at the operator insertion. Constraining the deficit angle to be less than π leads to maximum value h_{max} of order $\frac{1}{G_N}$ for allowed operators. Using the relation (2.9) between the weight and the mass gives a bound on the mass

$$m_{max} \sim \frac{1}{\ell G_N} \,. \tag{2.20}$$

In order to get a precise value for m_{max} one must specify the gravitational dynamics. The values will be numerically different for the pure Liouville theory and the dilaton-modified theories encountered in string compactification. For the moment we simply remark that a bound of the form (2.20) is in qualitative agreement with the stringy exclusion principle of [4], and might provide a generalization of the principle to non-BPS states.

2.4 Remarks

In this subsection we make several remarks on the preceding results.

Roughly speaking one might attempt to decompose the central charge as

$$0 = c_{total} = c_{gravity} + c_{matter} + c_{ghosts} \,. \tag{2.21}$$

The asymptotic growth of physical states, which is related to black hole entropy, should be determined by c_{matter}. Naively c_{matter} could be computed from the two-point function of the matter stress tensor at short distances in the bulk. One might expect that every field ϕ is an independent quantum field and contributes to c_{matter} because the mass term does not affect the two-point function at sufficiently short distances. However this cannot be correct since there can be an infinite number of such fields *e.g.* from Kaluza-Klein compactification or massive string modes. A bound of the form (2.20) alleviates, but does not seem in general to eliminate the problem. Presumably there are further quantum relations between the fields (*e.g.* of the kind encountered in [4]) which become very significant at short distances.

An alternate and possibly more efficient method of determining c_{matter} is from the two point function of the stress tensor on the boundary. The relation (2.17) implies that ϕ does not contribute to the two-point function of the stress tensor at the boundary if it has weight $h > 1$. Hence we need only consider zero or negative $(\text{mass})^2$ fields in such a computation.

It is interesting to contrast the preceding AdS_2 analysis with the AdS_3 analysis of [3] and [4]. In that context the states are also described by a conformal field theory, but it lives on the boundary of AdS_3 [2], whereas in the present AdS_2 context the conformal field theory is in the bulk of spacetime. The analysis of the scalar wave equation in [4] paralleled that of the previous paragraph. However it employed

JHEP01(1999)008

$SL(2,R)_L \times SL(2,R)_R$ rather than just a single $SL(2,R)$. Furthermore the form of the differential operators used in the AdS_3 case was not the familiar one of conformal field theory: the operators acted on a three dimensional space. The action reduces to the familiar form only on the boundary of AdS_3. In the AdS_2 context the connection with conformal field theory is more direct.

3. $AdS_3 \to AdS_2$

IIB string theory on $K3$ (similar statements pertain to T^4) has black string solutions characterized by D-onebrane (D-fivebrane) charge Q_1 (Q_5) and momentum density $\frac{n}{R^2}$. These objects have a dual description as a $c = 6Q_1Q_5$ conformal field theory whose target space is the symmetric product of Q_1Q_5 copies of $K3$ [20]. The near-horizon geometry of the spacetime solutions is $AdS_3 \times S^3 \times K3$. Compactifying the black string on a circle to yield a five-dimensional black hole corresponds to S^1 compactification of AdS_3 [21]. In this section we describe the appearance of AdS_2 in the very-near-horizon limit of this compactification.

3.1 S^1 compactification

We begin with the near-horizon AdS_3 geometry for the extremal five-dimensional black hole with charges (Q_1, Q_5, n)

$$ds^2 = \ell^2 T^2 \left(dx^5 + \frac{dt}{R}\right)^2 + \frac{U^2}{\ell^2}\left(\left(Rdx^5\right)^2 - dt^2\right) + \ell^2 \frac{(dU)^2}{U^2}\,. \tag{3.1}$$

t here is the asymptotic time coordinate in the full spacetime solution, while the normalization of x^5 is such that

$$x^5 \sim x^5 + 2\pi\,, \tag{3.2}$$

and all factors of the asymptotic S^1 radius R are explicit in the metric. We further define

$$\begin{aligned} T &= \sqrt{\frac{n}{k}}\,, \\ \ell^4 &= g_6^2 k\,, \\ k &= Q_1 Q_5\,, \end{aligned} \tag{3.3}$$

where g_6 is the six-dimensional string coupling and the dimensionless quantity T is related to the temperature T_L of the left-moving modes of the symmetric product conformal field theory by $\pi RT = T_L$. The classical supergravity limit is $g_6 \to 0$ with $g_6^2 n$, $g_6 Q_1$ and $g_6 Q_5$ fixed and large.

To get from AdS_3 to AdS_2 we decompose

$$ds_3^2 = ds_2^2 + \ell^2 e^{2\psi}(dx^5)^2 + \ell^2 e^{2\psi} A_t^2 dt^2 + 2\ell^2 e^{2\psi} A_t dt dx^5\,, \tag{3.4}$$

with

$$e^{2\psi} = T^2 + \frac{R^2U^2}{\ell^4}\,,$$
$$A_t = \frac{T^2}{R}e^{-2\psi}\,. \tag{3.5}$$

This yields

$$ds_2^2 = -\frac{R^2U^4}{\ell^6T^2+\ell^2R^2U^2}dt^2 + \frac{\ell^2}{U^2}dU^2\,. \tag{3.6}$$

3.2 The very-near-horizon region

The metric (3.6) reduces to the AdS_2 metric in the "very-near horizon" limit $\lambda \equiv \frac{U^2R^2k}{\ell^4 n} \to 0$, in which the S^1 radius ψ is constant. Radii of order U_0 correspond to excitation energies μ above extremality (see section 3.4 below) of order $\mu \sim \frac{k^2R^3U_0^4}{n\ell^8} \sim \lambda^2\frac{n}{R}$. Hence for $\lambda \to 0$ one is below the black hole mass gap [27]. The usual excited states are absent in this region. In terms of the dual symmetric-product conformal field theory this means the number of right-moving excitations obeys $n_R = 0$. The ground states (characterized by n_L) remain and should be described in this limit.

In addition to the ground state there is the possibility of a continuum of arbitrarily low-lying modes corresponding to the separation of the black hole into separate pieces. These modes are ignored in most discussions of black hole dynamics. In terms of the dual conformal field theory these are excitations of the Coulumb branch. This is an interesting subject and perhaps ultimately essential for a complete understanding of AdS_2. However at present we do not understand the description of these modes in detail and simply ignore them for the rest of the paper.

One may also consider the region of U^2 much, but not infinitely smaller than $\frac{\ell^4 n}{R^2 k}$, in which case λ is small but nonzero and the metric is nearly AdS_2. AdS_2/CFT_1 duality should yield approximate relations in this region in the spirit of [28]. This corresponds to excitation energies μ much less than $\frac{n}{R}$, or $n_R \ll n_L$. This is the regime of interest for the purpose of analyzing the near-extremal entropy.

In the very-near-horizon limit (3.6) reduces to

$$ds^2 = -\frac{\ell^2}{(t^+-t^-)^2}dt^+dt^-\,, \tag{3.7}$$

with $t^\pm = t \pm \frac{\ell^4 T}{2RU^2}$. The length of the internal circle is

$$\ell e^\psi = \ell\sqrt{\frac{n}{k}}\,. \tag{3.8}$$

The Kaluza-Klein gauge field strength is

$$F_{+-} = \partial_+A_- - \partial_-A_+ = -\frac{1}{T(t^+-t^-)^2} = \frac{2\epsilon_{+-}}{\ell^2}\sqrt{\frac{k}{n}}\,. \tag{3.9}$$

Note that the asymptotic modulus R, which is a fixed scalar, drops out of the very-near-horizon geometry.

The very-near-horizon geometry can be described directly as a quotient of AdS_3 without a limiting procedure. It is useful to transform to Poincare coordinates on AdS_3 given by

$$\begin{aligned} w^+ &= \frac{R}{2T} e^{2T\left(x^5+\frac{t}{R}\right)}\,, \\ w^- &= (Rx^5 - t) - \frac{\ell^4 T}{RU^2}\,, \\ y &= \frac{\ell^2}{U} e^{T\left(x^5+\frac{t}{R}\right)}\,, \end{aligned} \tag{3.10}$$

in which the metric becomes

$$ds_3^2 = \frac{\ell^2}{y^2}(dw^+ dw^- + dy^2)\,. \tag{3.11}$$

The identification $x^5 \sim x^5 + 2\pi$ corresponds to

$$\begin{aligned} w^+ &\sim e^{4\pi T} w^+\,, \\ w^- &\sim w^- + 2\pi R\,, \\ y &\sim e^{2\pi T} y\,. \end{aligned} \tag{3.12}$$

The very-near-horizon limit can be reached by the diagonal $SL(2,R)_L \times SL(2,R)_R$ transformation

$$\begin{aligned} w^\pm &\to \frac{w^\pm}{\lambda'}\,, \\ y &\to \frac{y}{\lambda'}\,, \end{aligned} \tag{3.13}$$

for $\lambda' \to 0$, since the horizon is at $y = \infty$. This transformation takes $R \to \lambda' R$, so the limit is equivalent to $R \to 0$. In this case (3.12) is a purely $SL(2,R)_L$ transformation, and contains no $SL(2,R)_R$ action. Hence we recover the construction of [29] of the near-horizon AdS_2 geometry as an $SL(2,R)_L$ quotient of AdS_3. As we shall see later, this leads to a somewhat subtle relation between the unbroken $SL(2,R)_R$ isometries of AdS_3 and the $SL(2,R)$ isometries of AdS_2.

The two-dimensional AdS_2 geometry resulting from the quotient (3.12) with R set to zero is most simply expressed in terms of[8]

$$\begin{aligned} w^- &= t^-\,, \\ w^+ &= \frac{1}{2T} e^{2Tu}\,, \\ y &= \sqrt{\frac{t^+ - t^-}{2T}}\, e^{Tu}\,. \end{aligned} \tag{3.14}$$

[8] Alternately, $u = x^5 + (t/R) + (\ln R/2T)$, $t^\pm = Rx^5 - t \pm (\ell^4 T/RU^2)$.

The inverse transformation is

$$
\begin{aligned}
t^- &= w^- \,, \\
t^+ &= \frac{y^2}{w^+} + w^- \,, \\
u &= \frac{1}{2T} \ln 2Tw^+ \,.
\end{aligned}
\tag{3.15}
$$

One then has

$$
ds_3^2 = \frac{\ell^2 (dt^+ - dt^-)^2}{4(t^+ - t^-)^2} + \frac{\ell^2 T du (dt^+ + dt^-)}{(t^+ - t^-)} + \ell^2 T^2 du^2 \,. \tag{3.16}
$$

Imposing the null identification

$$
u \sim u + 2\pi \tag{3.17}
$$

and using (3.4) gives

$$
\begin{aligned}
e^{2\psi} &= T^2 \,, \\
A_\pm &= \frac{1}{2T(t^+ - t^-)} \\
ds_2^2 &= \frac{\ell^2 dt^+ dt^-}{(t^+ - t^-)^2}
\end{aligned}
\tag{3.18}
$$

The coordinates $t^\pm$ cover only the region outside the horizon, as illustrated. Coordinates which cover all of (the universal cover of) AdS_2 are[9] $t^\pm = \tan u^\pm$. The metric is

$$
ds^2 = -\frac{\ell^2 du^+ du^-}{\sin^2 (u^+ - u^-)} \tag{3.19}
$$

$\sigma = (u^+ - u^-)$ runs from 0 to π while $\tau = u^+ + u^-$ is unrestricted. The horizons are at $u^\pm = \pm\frac{\pi}{2}$.

Note that AdS_2 has two timelike boundaries $\sigma = 0, \pi$, one inside and one outside the horizon. The electric field strength (3.9) can be viewed as arising from charges $\frac{k\ell^2}{2} e^{3\psi} \epsilon^{+-} F_{+-} = \pm n$ fixed at the boundaries.

3.3 The dual description

String theory on AdS_3 has a dual description as a $c = 6k$ (deformed) symmetric product conformal field theory. This theory lives on the cylindrical boundary of AdS_3 with $y = 0$. This boundary has null coordinates $w^\pm$. The action of the identification (3.12) in the very near horizon limit for which $R \to 0$ is simply a shift of $\ln w^+$, with no action on w^-. Hence the dual representation of the very-near horizon string theory is a DLCQ conformal field theory. This suggests a connection with the matrix model [30, 31].

[9] Alternately, for $\tau = u^+ + u^-$, $\quad \mathrm{sech}\rho = \sin(u^+ - u^-)$, $\quad 4ds^2 = -\ell^2 \cosh^2 \rho d\tau^2 + \ell^2 d\rho^2$.

We note that the one dimension of the $D = 1$ conformal field theory dual to AdS_2 string theory is a null (w^-) coordinate. This provides a natural framework for the action of the Virasoro algebra, whose existence was argued for on general grounds in section 2. On the other hand the general arguments of [1, 25] suggest a formulation of the dual theory as a conformal quantum mechanics on the two timelike boundaries of AdS_2. It would be interesting to understand this formulation in detail.

It is intriguing to observe that the general $AdS_D \leftrightarrow CFT_{D-1}$ duality boils down in this $D = 2$ context to a duality between two conformal field theories, one with $c = 0$ on the strip and another with $c = 6k$ on the DLCQ cylinder. Perhaps this $D = 2$ duality can be understood in more conventional field theoretic terms.

3.4 Two-dimensional effective action

The gravity sector in AdS_3 is governed by the action

$$S_3 = \frac{k}{4\pi\ell} \int d^3x \sqrt{-g} \left(R + \frac{2}{\ell^2} \right) . \tag{3.20}$$

Additional fields suppressed here include the $SU(2)_L \times SU(2)_R$ level k Chern-Simon gauge fields. (3.20) reduces to a two-dimensional metric, scalar and $U(1)$ gauge field governed by[10]

$$S_2 = \frac{k}{2} \int d^2x \sqrt{-g} \left(e^{\psi} \left(R + \frac{2}{\ell^2} \right) - \frac{\ell^2}{4} e^{3\psi} F^2 \right) . \tag{3.21}$$

This action has a solution with $R = -\frac{8}{\ell^2}$ and constant ψ and F given by (3.8)–(3.9), corresponding to the extremal black hole. Additionally there are near extremal solutions given by

$$\begin{aligned} ds_2^2 &= -\frac{U^2(U^2 - U_0^2)}{\ell^2(U^2 + U_0^2 \sinh^2 \sigma)} dt^2 + \frac{\ell^2}{U^2 - U_0^2} dU^2 , \\ e^{2\psi} &= \frac{U^2 + U_0^2 \sinh^2 \sigma}{\ell^4} , \\ A_t &= \frac{U_0^2 \sinh 2\sigma}{2(U^2 + U_0^2 \sinh^2 \sigma)} , \end{aligned} \tag{3.22}$$

with U_0 and σ defined by

$$\begin{aligned} n &= \frac{kU_0^2 \sinh 2\sigma}{2\ell^4} , \\ \mu &= \frac{kU_0^2 e^{-2\sigma}}{2\ell^4} . \end{aligned} \tag{3.23}$$

n is related to the momentum around the S^1 while μ is the energy above extremality.

[10] In conformal gauge, $S_2 = \frac{k}{2} \int d^2x \left(4e^{\psi} \partial_- \partial_+ \rho + \frac{1}{\ell^2} e^{\psi + 2\rho} + \ell^2 e^{3\psi - 2\rho} F_{+-}^2 \right)$. For fixed charge the last term is $\frac{k}{2} \int d^2x e^{-3\psi + 2\rho} \left(\frac{n}{\ell k} \right)^2$.

Note that for large U and nonzero μ, the radius ψ blows up. This means that asymptotically one effectively returns to AdS_3.[11] Hence the near extremal solution can be understood in the AdS_2 context only for sufficiently small U and U_0. This requires a small energy μ. To leading non-trivial order for such small μ and fixed n the entropy is

$$S_{BH} = 2\pi k e^{\psi(U_0)} = 2\pi\sqrt{kn} + \pi\sqrt{2k\mu}\,. \tag{3.24}$$

4. Symmetries

In this section we consider the symmetries of AdS_2 string theory and its relation to those of AdS_3.

4.1 The unbroken $SL(2,R)$

The quotient (3.12) in the very-near-horizon limit $R \to 0$ leaves intact the full $SL(2,R)_R$ isometry group of AdS_3. One expects this group is related to the $SL(2,R)$ isometry group of AdS_2. Using the coordinate transformations (3.15) and the AdS_3 isometries one finds that the AdS_3 and AdS_2 isometries are related by

$$\begin{aligned}
H_{-1} &= i\frac{\partial}{\partial w^-} = i\left(\frac{\partial}{\partial t^+} + \frac{\partial}{\partial t^-}\right)\,,\\
H_0 &= i\left(w^-\frac{\partial}{\partial w^-} + \frac{1}{2}y\frac{\partial}{\partial y}\right) = i\left(t^+\frac{\partial}{\partial t^+} + t^-\frac{\partial}{\partial t^-}\right)\,,\\
H_1 &= i\left((w^-)^2\frac{\partial}{\partial w^-} + w^- y\frac{\partial}{\partial y} - y^2\frac{\partial}{\partial w^+}\right)\\
&= i\left(t^{+2}\frac{\partial}{\partial t^+} + t^{-2}\frac{\partial}{\partial t^-} - \frac{t^+ - t^-}{2T}\frac{\partial}{\partial u}\right)\,.
\end{aligned} \tag{4.1}$$

We see that the H_{-1} and H_0 generators are identified for AdS_2 and AdS_3, but there is a shift in H_1. Since u is a coordinate on the compactified S^1, the extra term can be interpreted as a gauge transformations. Hence the $SL(2,R)_R$ AdS_3 isometries reduce to the $SL(2,R)$ AdS_2 isometry plus a gauge transformation.

Given such a relation between the AdS_2 and AdS_3 isometry groups one naturally expects a relation between the full conformal groups. To find this we note that the AdS_2 action in (4.1) can be written as

$$H_j = i\left(\epsilon_j^+\partial_+ - \frac{\partial_+\epsilon_j^+}{4T}\partial_u + \epsilon_j^-\partial_- + \frac{\partial_-\epsilon_j^-}{4T}\partial_u\right)\,, \tag{4.2}$$

with

$$\epsilon_j^\pm = (t^\pm)^{j+1}\,, \tag{4.3}$$

[11] ψ is an $m^2\ell^2 = 8$ fixed scalar which according to (2.9) corresponds to an irrelevant $h = 2$ perturbation of the AdS_2 conformal field theory. Hence exciting this field corresponds to changing the AdS_2 boundary conditions.

for $j = 0, \pm 1$. For general integral values of $-\infty < j < \infty$ one easily finds that the H_k obey the classical Lie bracket relation

$$\{H_j, H_k\}_{L.B.} = i(j-k)H_{j+k}. \tag{4.4}$$

One also finds for all values of j using the coordinate transformation (3.15) that the action of H_k near the boundary $y = 0$ of AdS_3 is

$$H_k = i\left(\zeta_k^- \frac{\partial}{\partial w^-} + \partial_- \zeta_k^- \frac{y}{2}\frac{\partial}{\partial y} - \partial_-^2 \zeta_k^- \frac{y^2}{2}\frac{\partial}{\partial w^+}\right)\left(1 + \mathcal{O}(y^2)\right), \tag{4.5}$$

where $\zeta_k^- = (w^-)^{k+1}$. This expression — including the subleading terms indicated — is the same as that given in [2] for diffeomorphisms preserving the AdS_3 boundary conditions. Hence the H_k generate the right-moving AdS_3 conformal transformations.

The precise form for all values of y of the generators (4.1) of the SL(2,R) subgroup is fixed by demanding that they are Killing vectors of the AdS_3 metric. The other generators are identified only by their action at infinity and in general have no preferred form in the interior. Equation (4.2) for general j is a convenient choice for our purposes.

Let us now assume that we can define a quantum theory on AdS_2 with operators T and U corresponding to the full $c = 0$ stress tensor and the appropriately normalized conformal current associated to the Kaluza-Klein $U(1)$ gauge symmetry. Then the diffeomorphisms (4.2) are generated by moments of the twisted stress tensor

$$\tilde{T}_{\pm\pm} = T_{\pm\pm} \mp \partial_\pm U_\pm\,. \tag{4.6}$$

Twisted Virasoro generators $\tilde{L}_n$ can be defined in terms of $\tilde{T}$. This yields the general relation between the right Virasoro generators $L_n^{(3)}$ of AdS_3 and those of AdS_2:

$$L_n^{(3)} = \tilde{L}_n = L_n + (n+1)U_n. \tag{4.7}$$

Energy conservation $T_{--} = T_{++}$ as well as $\tilde{T}_{--} = \tilde{T}_{++}$ on the boundary requires that the current U obeys Dirichlet boundary conditions:

$$U_+(\sigma) = -U_-(\sigma)\,, \qquad \text{for } \sigma = 0, \pi\,, \tag{4.8}$$

with respect to which the modes of U in (4.7) are defined. These boundary conditions correspond to fixed total charge or fixed electric flux (3.9).[12] The level of the $U(1)$ current algebra can be deduced from (4.7) and the assumption that the AdS_3 central charge is $c = 6k$ while the AdS_2 central charge is $c = 0$. One finds

$$[U_m, U_n] = -\frac{km}{2}\delta_{m+n}\,. \tag{4.9}$$

It would be of interest to compute this level directly.

[12]The charge density U_t does not include the fixed boundary charges $\pm n$.

The assumption that a suitable quantum theory can be defined on AdS_2 is perhaps not as benign as it sounds. In order to do so one must specify boundary conditions on all the fields, and perhaps also understand the ground state entropy in (3.24). The problem is analogous to that of finding the Hilbert space in DLCQ, and similar subtleties may be encountered. A further subtlety is the possibility of singleton degrees of freedom which are confined to the boundary of the strip but could interact with the bulk degrees of freedom. We note that the AdS_2 Hilbert space is not the same as the AdS_3 Hilbert space: they should agree only after the Virasoro constraints are imposed on the former. Indeed if the two-dimensional L_ns are translated back to AdS_3 they do not generate diffeomorphisms of the form (4.5) — the last term is missing — and hence create states which fail to obey the AdS_3 boundary conditions of [2] by a coordinate transformation. Hence the unconstrained AdS_2 Hilbert space is larger than AdS_3 by spurious states, which are hopefully eliminated by constraints.

In addition to the $U(1)$ gauge symmetry arising from the S^1 isometry there is an $SU(2)_L \times SU(2)_R$ gauge symmmetry from the isometries of S^3. This appears to lead to the large $N = 4$ A_γ algebra [32]. Since $c = 0$ it is the degenerate case $\gamma = 0$. This algebra is known [32] to contain the standard $N = 4$ algebra as a subalgebra. The relation between the Virasoro generators is exactly (4.7).

The twisting (4.7) of the stress tensor has also appeared as a central formula in the study of topological field theory [22]. In that context one begins with a conventional supersymmetric conformal field theory with a unitary Hilbert space. This corresponds to the AdS_3 conformal field theory in the present discussion. The stress tensor is then twisted to shift the central charge to zero exactly as in (4.7). Once the central charge is zero, the theory can be reinterpreted as a theory of gravity. Since the partition function for gravity is independent of a choice of background metric this is a topological field theory.

4.2 Charged Kaluza-Klein modes

AdS_3 string theory has a tower of scalar fields in short BPS multiplets of various masses as described in [4, 5, 6]. For every scalar field of mass m on AdS_3 there is a corresponding scalar field of mass m on AdS_2 obtained from the constant mode on S^1. For each of these fields there is a highest weight state with weight given by (2.9) as $h = \frac{1}{2}\left(1 + \sqrt{1 + m^2\ell^2}\right)$, and a corresponding primary operator in the boundary conformal field theory.

The full Kaluza-Klein tower of fields ϕ_q on AdS_2 arises from S^1 modes e^{iqu}, where q is an integer. These states have charge q and mass

$$m_q^2 = m^2 + \frac{q^2}{\ell^2 T^2}\,, \tag{4.10}$$

where m is the AdS_3 mass of the scalar field. We wish to find the weight of the highest weight state created by ϕ_q, following the $q = 0$ analysis of section 2. The ϕ_q

wave equation is

$$\mathcal{D}^2\phi_q = m_q^2\phi_q\,. \tag{4.11}$$

Using $\mathcal{D} = \nabla - iqA$, the conformal gauge metric (2.3) on the strip, equation (4.10) and the gauge

$$A_- = A_+ = \frac{\cos(u^+ - u^-)}{2T\sin(u^+ - u^-)}\,, \tag{4.12}$$

equation (4.11) can be rewritten as

$$\Big(-4\sin^2(u^+ - u^-)\partial_+\partial_- + \frac{iq}{T}\sin 2(u^+ - u^-)(\partial_+ + \partial_-) - \\ -\frac{q^2}{T^2}\sin^2(u^+ - u^-)\Big)\,\phi_q = m^2\phi_q\,. \tag{4.13}$$

In terms of the twisted Virasoro generators on the strip

$$\tilde{L}_n = \frac{i}{2}\left(e^{2niu^+}\partial_+ + e^{2niu^-}\partial_-\right) + \frac{iqn}{4T}\left(e^{2niu^+} - e^{2niu^-}\right). \tag{4.14}$$

equation (4.13) has the simple form

$$-4\left(-\tilde{L}_0\left(\tilde{L}_0 - 1\right) + \tilde{L}_{-1}\tilde{L}_1\right)\phi_q = m^2\ell^2\phi_q\,. \tag{4.15}$$

It follows that the highest weight solutions obeying $\tilde{L}_1\phi_{h,q} = 0$ have a weight h which is independent of q and given by (2.9). The wave function is

$$\phi_{h,q} = \left(e^{-2iu^+} - e^{-2iu^-}\right)^h e^{-\frac{q}{2T}(u^+ - u^-)}\,. \tag{4.16}$$

We note that the boundary operators $(\partial_\sigma)^k\phi_q(0,\tau)$ vanish for $k < h$ independently of q. To summarize, for every field on AdS_3 of mass m and associated weight h, there is an tower of perturbative[13] fields labeled by the Kaluza-Klein charge q. These have q dependent masses (4.10) but q-independent weights (2.9). In particular all the charged Kaluza-Klein modes of the $m^2 = 0$ scalars on AdS_3 are $h = 1$ scalars on AdS_2. These will all have non-vanishing boundary stress tensor and contribute to boundary computation of the central charge.

Acknowledgments

I would like to thank T. Banks, R. Gopakumar, J.Harvey, J. Maldacena, E. Martinec, G. Moore, S. Shenker, C. Vafa and A. Volovich for useful conversations. I am grateful to the Institute for Theoretical Physics at Santa Barbara and the Aspen Center for physics for hospitality while parts of this work were carried out. The work was supported in part by DOE grant DE-FG02-91ER40654.

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